

ON KEEN WEAKLY REDUCIBLE HEEGAARD SPLITTINGS

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ABSTRACT. A Heegaard splitting which admits a unique pair of disjoint compression disks on distinct sides is said to be keen weakly reducible. This paper provides an construction of keen weakly reducible Heegaard splittings of arbitrary genus except 2. Furthermore, critical Heegaard splittings may yield if we change some conditions.

1. INTRODUCTION

Let M be an oriented compact 3-manifold. If there exists a closed surface S which cuts M into two compression bodies V and W , such that $S = \partial_+ V = \partial_+ W$, then $M = V \cup_S W$ is called a Heegaard spitting of M and S is called a Heegaard surface of M . It is well known that every compact connected orientable 3-manifold admits a Heegaard splitting.

A Heegaard splitting $M = V \cup_S W$ is said to be reducible if there are two essential disks $D \subset V$ and $E \subset W$ such that $\partial D = \partial E$. In other words, there exists a 2-sphere which intersects S in an essential curve; Otherwise, $M = V \cup_S W$ is said to be irreducible.

A Heegaard splitting $M = V \cup_S W$ is said to be weakly reducible[6] if there are two essential disks $D \subset V$ and $E \subset W$ such that $\partial D \cap \partial E = \emptyset$; Otherwise, it is said to be strongly irreducible.

Let S be a closed surface whose genus is at least 2. The distance between two essential simple closed curves α and β in S , denoted by $d_S(\alpha, \beta)$, is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ in S such that α_{i-1} is disjoint from α_i for $1 \leq i \leq n$. Let A and B be two sets of essential simple closed curves in S . The distance between A and B , which is denoted by $d_S(A, B)$, is defined to be $\min\{d_S(x, y) | x \in A, y \in B\}$.

If D and E are two compression disks on distinct sides of S , sometimes $d_S(\partial D, \partial E)$ is denoted simply by $d_S(D, E)$. For examples, suppose

2010 *Mathematics Subject Classification.* 57N10, 57M27.

Key words and phrases. 3-manifolds, Heegaard surfaces, weakly reducible, topologically minimal surfaces .

This research is supported by grants (No.11401069 and No.11671064) of NSFC and Fundamental Research Funds for the Central Universities (No. 3132015231).

that $g(S) > 1$ and ∂D intersects ∂E in one point then $d_S(D, E) = 2$, since $\partial[N(\partial D \cup \partial E)] \subset S$ is essential and disjoint from $\partial D \cup \partial E$.

The distance of the Heegaard splitting $V \cup_S W$ is defined to be $d(S) = d_S(\mathcal{D}_V, \mathcal{D}_W)$ where \mathcal{D}_V and \mathcal{D}_W are sets of essential disks in V and W , respectively. $d(S)$ was first defined by Hempel, see [9].

$M = V \cup_S W$ is said to be a keen Heegaard splitting, if its distance is realized by a unique pair of elements \mathcal{D}_V and \mathcal{D}_W . For any integers $n \geq 2$ and $g \geq 3$, there exists a strongly keen Heegaard splitting of genus g whose distance is n . For more details, see [10].

The distance of $M = V \cup_S W$ is equal to 1 if and only if it is irreducible and weakly reducible. By definition, $M = V \cup_S W$ is said to be a keen weakly reducible Heegaard splitting, if there exists only one pair of disjoint compression disks on opposite sides of the Heegaard surface.

The simplest example of keen weakly reducible Heegaard splittings is the genus 1 Heegaard splitting of $S^2 \times S^1$ which is reducible. Moreover, it is well known that there is no keen weakly reducible Heegaard splitting of genus 2. In the following statement, we assume that the the genus of the Heegaard surface is at least 3.

Suppose that $M = V \cup_S W$ is a keen weakly reducible Heegaard splitting, $D_0 \subset V$ and $E_0 \subset W$ are disjoint compression disks. It is easy to observe that ∂D_0 and ∂E_0 are not isotopic, moreover, each is non-separating on S , and $\partial D_0 \cup \partial E_0$ is separating on S . It follows that $V \cup_S W$ is irreducible and unstabilized.

Let F be a properly embedded, separating surface with no torus components in a compact, orientable, irreducible 3-manifold M . Then the disk complex of F , denoted by $\mathcal{D}(F)$, is defined as follows: Vertices of $\mathcal{D}(F)$ are isotopy classes of compression disks for F , and a set of $m + 1$ vertices forms an m -simplex if there are representatives for each that are pairwise disjoint.

David Bachman explored the information which is contained in the topology of $\mathcal{D}(F)$ by defining the topological index of F [5]. If $\mathcal{D}(F)$ is non-empty then the topological index of F is the smallest n such that $\pi_{n-1}(\mathcal{D}(F))$ is non-trivial. If $\mathcal{D}(F)$ is empty then F will have topological index 0. If F has a well-defined topological index (i.e. $\mathcal{D}(F) = \emptyset$ or non-contractible) then we will say that F is a topologically minimal surface.

This raises a question that which surfaces are topologically minimal surfaces, moreover, if a surface is topologically minimal, we may ask what the topological index is. By definition, F has topological index 0 if and only if it is incompressible, and has topological index 1 if and only if it is strongly irreducible. So we only need to consider weakly reducible surfaces. If a weakly reducible surface is topologically minimal, the topological index is

at least 2. Index 2 topologically minimal surface are called critical surfaces which are also defined by David Bachman, see[1]and[3].

If a weakly reducible Heegaard splitting is keen, the disk complex of the Heegaard surface is quite simple. By McCullough in[12], who showed that the disk complex of the boundary of a handlebody is contractible, the disk complex of the Heegaard surface is obtained by attaching these two together with a single edge. That will just create a larger contractible complex. Thus we have the following proposition.

Proposition 1.1. *Suppose that $V \cup_S W$ is a keen weakly reducible Heegaard splitting, then the disk complex of the Heegaard surface S is contractible, in other words, S is not topologically minimal.*

In [8], The author proved that some self-amalgamated Heegaard surfaces are keen. We remark that the genera of all the Heegaard surfaces given in [8] are odd, and the main theorem contains strict assumptions. In this paper, we will provide an explicit construction of keen weakly reducible Heegaard splittings of arbitrary genus. Such splittings admit a “symmetrical structure” and there is a lot of flexibility in doing so. Moreover, we may construct critical Heegaard splittings if we change some conditions.

2. PRELIMINARIES

Let V^* be a handlebody. Denote its boundary by S^* . Let V be the handlebody obtained by attaching one 1-handle $D_0 \times I$ to V^* along a pair of disjoint disks $D_1, D_2 \subset S^*$, where D_0 is a disk corresponding to the 1-handle. Thus $g(V) = g(V^*) + 1$. Recall \mathcal{D}_V is defined to be the set of compression disks of V . There is a natural partition of \mathcal{D}_V induced by D_0 up to isotopy, as follows:

$$\mathcal{D}_V = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

$$\mathcal{D}_0 = \{D_0\}.$$

$$\mathcal{D}_1 = \{D \mid D \cap D_0 = \emptyset; D \neq D_0; D \text{ is inessential in } V^*\}.$$

$$\mathcal{D}_2 = \{D \mid D \cap D_0 = \emptyset; D \text{ is essential in } V^*\}.$$

$$\mathcal{D}_3 = \{D \mid D \cap D_0 \neq \emptyset\}.$$

Let D be a compression disk of V . Then D belongs to one and only one of the four subsets. If $D \in \mathcal{D}_1$, then ∂D , ∂D_1 and ∂D_2 co-bound a pair of pants on S^* . It follows that D is isotopic to a band-sum of D_1, D_2 along an arc and D cuts V into V^* and a solid torus. $D \in \mathcal{D}_2$ if and only if $D \in \mathcal{D}_{V^*}$, obviously. See figure 1.

Now consider $D \in \mathcal{D}_3$. Suppose D is an essential disk in V such that $D \cap D_0 \neq \emptyset$. Furthermore, D is isotoped in V such that $|D \cap D_0|$ is minimal. Let S_1 be the surface $S^* - (\text{int}(D_1) \cup \text{int}(D_2))$. Then S_1 is a sub-surface of S^* with two boundary components ∂D_1 and ∂D_2 . By standard arguments, we have some observations as follows. See also[15].

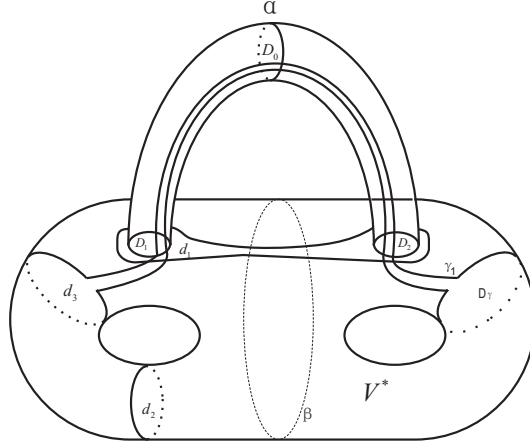


FIGURE 1. disks in the handlebody such that $d_i \subset \mathcal{D}_i$

Lemma 2.1.

- (1) Each component of $D \cap D_0$ is a properly embedded arc in both D and D_0 .
- (2) Each component of $\partial D \cap S_1$ is essential on S_1 .
- (3) Each component of $D \cap (\partial D_0 \times I)$ is an arc with its two end points lying in distinct boundary components of the annulus $\partial D_0 \times I$.

An essential arc γ in S_1 is called strongly essential if both boundary points lie in ∂D_i and γ is an essential arc on $S_1 \cup D_j$, where $\{i, j\} = \{1, 2\}$. Recall that $D \subset \mathcal{D}_3$. Let γ be an outermost component of $D \cap (D_1 \cup D_2)$ on D . This means that γ , together with an arc $\gamma_1 \subset \partial D$, bounds a sub-disk in D , say D_γ , such that $D_\gamma \cap (D_1 \cup D_2) = \gamma$. We call γ_1 an outermost arc related to γ and call D_γ an outermost disk related to γ .

Lemma 2.2. [15]

- (1) γ_1 , whose end points lie in one of D_1 and D_2 , is strongly essential in S_1 .
- (2) $D_\gamma \subset V^*$ and is essential in V^* .

3. CONSTRUCTION OF KEEN WEAKLY REDUCIBLE HEEGAARD SPLITTINGS

In this section, we will provide an explicit construction of keen weakly reducible Heegaard splittings of genus g .

Let V^* be a genus $(g - 1)$ handlebody. We choose an essential curve $\beta \subset \partial V^*$, such that β is separating on ∂V^* and $d_{\partial V^*}(\beta, \mathcal{D}_{V^*}) \geq 4$. Then we attach a 2-handle $E_0 \times I$ to V^* along β so that β bounds a disk E_0 . This yields a 3-manifold denoted by M_1 .

Next, we attach a 1-handle $D_0 \times I$ to M_1 such that the gluing disks denoted by D_1, D_2 are on distinct components of $\partial V^* - \beta$, where D_0 is a disk corresponding to the 1-handle and we denote ∂D_0 by α . The resulted 3-manifold denoted by M_2 . ∂M_2 is connected and $g(\partial M_2) = g - 1$.

Next, let W^* be another genus $(g - 1)$ handlebody. We glue W^* to M_2 via an orientation preserving homeomorphism $f : \partial W^* \rightarrow \partial M_2$ such that $d_{\partial W^*}(f(\alpha), \mathcal{D}_{W^*}) \geq 4$. Since α is separating on ∂M_2 , $f(\alpha)$ is separating on ∂W^* . Recall $D_1 \cup D_2 = V^* \cap (D_0 \times I)$. Similarly, $W^* \cap (E_0 \times I)$ are two disks, denoted by E_1, E_2 .

The result is a closed 3-manifold denoted by M . Denote $V = V^* \cup (D_0 \times I)$, $W = W^* \cup (E_0 \times I)$ and $S = V \cap W$. Then both V and W are genus g handlebodies and $M = V \cup_S W$ is a Heegaard splitting. α bounds the disk D_0 in V and β bounds the disk E_0 in W . Since the 2-handle $E_0 \times I$ is attached before the 1-handle $D_0 \times I$, $D_0 \cap E_0 = \alpha \cap \beta = \emptyset$. It follows that $M = V \cup_S W$ is weakly reducible.

Theorem 3.1. $M = V \cup_S W$ is a keen weakly reducible Heegaard splitting.

Proof. In order to show $V \cup_S W$ is keen, we firstly divide \mathcal{D}_V and \mathcal{D}_W as mentioned in Section 2:

$$\mathcal{D}_V = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

and

$$\mathcal{D}_W = \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2 \cup \mathcal{D}^3$$

such that:

$$\begin{aligned} \mathcal{D}_0 &= \{D_0\}. \\ \mathcal{D}_1 &= \{D \mid D \cap D_0 = \emptyset; D \neq D_0; D \text{ is inessential in } V^*\}. \\ \mathcal{D}_2 &= \{D \mid D \cap D_0 = \emptyset; D \text{ is essential in } V^*\} = \mathcal{D}_{V^*}. \\ \mathcal{D}_3 &= \{D \mid D \cap D_0 \neq \emptyset\}. \\ \mathcal{D}^0 &= \{E_0\}. \\ \mathcal{D}^1 &= \{E \mid E \cap E_0 = \emptyset; E \neq E_0; E \text{ is inessential in } W^*\}. \\ \mathcal{D}^2 &= \{E \mid E \cap E_0 = \emptyset; E \text{ is essential in } W^*\} = \mathcal{D}_{W^*}. \\ \mathcal{D}^3 &= \{E \mid E \cap E_0 \neq \emptyset\}. \end{aligned}$$

In order to show that (D_0, E_0) is the unique disjoint pair of compression disks on distinct sides of S , we will show that for each $D \in \mathcal{D}_i$, $E \in \mathcal{D}^j$, where $i, j = 0, 1, 2, 3$ and $(i, j) \neq (0, 0)$, $D \cap E = \partial D \cap \partial E \neq \emptyset$ holds.

Claim 3.2. For each $D \in \mathcal{D}_0$, $E \in \mathcal{D}^1$, $D \cap E \neq \emptyset$.

In this case, $D = D_0$ and E is a band-sum of two copies of E_0 along an arc. Recall that $\alpha = \partial D_0$ and $\beta = \partial E_0$ and the union separates S . α separates the two copies of E_0 . Any arc connected to the two copies intersects α . It follows that $D \cap E \neq \emptyset$.

Claim 3.3. For each $D \subset \mathcal{D}_0$, $E \subset \mathcal{D}^2$, $D \cap E \neq \emptyset$.

In this case, $D = D_0$ and E is essential both in handlebodies W and W^* . Notice that $W^* = \overline{W \setminus (E_0 \times I)}$. Recall that $d(f(\alpha), \mathcal{D}_{W^*}) \geq 3$, thus D_0 intersects each compression disk of W^* .

Remark 3.4. By the above two claims, if E is an essential disk in W which is not isotopic to E_0 and disjoint from E_0 , then $D_0 \cap E \neq \emptyset$. Similarly, if D is an essential disk in V which is not isotopic to D_0 and disjoint from D_0 , then $D \cap E_0 \neq \emptyset$.

In the following argument, we assume that $|D \cap D_0| + |E \cap E_0|$ is minimal in the isotopy classes of D and E .

Claim 3.5. For each $D \subset \mathcal{D}_0$, $E \subset \mathcal{D}^3$, $D \cap E \neq \emptyset$.

In this case, $D = D_0$ and $E \cap E_0 \neq \emptyset$. Assume that $D_0 \cap E = \emptyset$. By applying Lemma 2.2 to W and W^* , there is an outermost disk of E , say E_γ , which is essential in W^* . By Claim 3.3, $D_0 \cap E_\gamma \neq \emptyset$. Notice that $D_0 \cap E_\gamma \subset D_0 \cap E$. This contradicts the assumption $D_0 \cap E = \emptyset$.

Claim 3.6. For each $D \subset \mathcal{D}_1$, $E \subset \mathcal{D}^1$, $D \cap E \neq \emptyset$.

In this case, D is a band-sum of two copies of D_0 and E is a band-sum of two copies of E_0 . ∂D bounds a once-punctured torus T_D on S and ∂E bounds a once punctured torus T_E . Suppose to the contrary that $D \cap E = \emptyset$. First we assume that ∂E is isotopic to ∂D . In this case, since $g(S) > 2$, we have $T_D = T_E$. It follows that α and β are isotopic because $\alpha \cap \beta = \emptyset$, a contradiction. Hence $\partial E \neq \partial D$. In this case $T_D \cap T_E = \emptyset$ and $\alpha \cup \beta$ does not separate the Heegaard surface, a contradiction.

Claim 3.7. For each $D \subset \mathcal{D}_1$, $E \subset \mathcal{D}^2$, $D \cap E \neq \emptyset$.

In this case, D is a band-sum of two copies of D_0 and E is an essential disk in W^* . ∂D bounds a once-punctured torus T_D on S and $\alpha \subset T_D$. Suppose to the contrary that $D \cap E = \emptyset$. If $\partial E \not\subset T_D$ then $E \cap D_0 = \emptyset$ which contradicts Claim 3.3. Thus $\partial E \subset T_D \subset S$. Moreover, $\partial E \subset N(\alpha) \subset S$ since $E \cap E_0 = \emptyset$. In this case ∂E is isotopic to α which implies that $d_{\partial W^*}(f(\alpha), \mathcal{D}_{W^*}) = 0$, a contradiction.

Claim 3.8. For each $D \subset \mathcal{D}_1$, $E \subset \mathcal{D}^3$, $D \cap E \neq \emptyset$.

In this case $E \cap E_0 \neq \emptyset$. By applying Lemma 2.2 to W and W^* , there is an outermost arc, say γ , whose two endpoints lie in one of ∂E_1 and ∂E_2 . Furthermore, there is an outermost disk of E , say E_γ , such that E_γ is essential in W^* . Without loss of generality, we assume that $E_\gamma \cap E_1$ is an arc denoted by e_1 , that is to say, $\partial \gamma \subset \partial E_1$.

D is a band-sum of two copies of D_0 and the two copies lie distinct sides of β . Hence there is a sub-arc γ_1 of ∂D such that $\partial\gamma_1 \subset \partial E_2$. γ_1 , together with a sub-arc of ∂E_2 , say e_2 , bounds a disk isotopic to D_0 .

$D \cap E = \emptyset$ implies that $\gamma \cap \gamma_1 = \emptyset$, thus $(\gamma \cup e_1) \cap (\gamma_1 \cup e_2) = \emptyset$ which means that $E_\gamma \cap D_0 = \emptyset$. This contradicts Claim 3.3.

Claim 3.9. For each $D \subset \mathcal{D}_2$, $E \subset \mathcal{D}^2$, $D \cap E \neq \emptyset$.

In this case, D is essential in V^* and E is essential in W^* . Suppose to the contrary that $D \cap E = \emptyset$. We remark that $\partial D \cap \beta \neq \emptyset$ and β separates ∂V^* . Hence there is a sub-arc γ_1 of ∂D such that $\partial\gamma_1 \subset \partial E_1$ and γ_1 together with an arc of E_1 forms an essential closed curve γ on ∂W^* . Moreover, $D_0 \cap D = \emptyset$ and $D_0 \cap \beta = \emptyset$ mean that $D_0 \cap \gamma = \emptyset$. $E \cap D = \emptyset$ and $E \cap \beta = \emptyset$ mean that $E \cap \gamma = \emptyset$. Hence $d_{\partial W^*}(f(\alpha), \mathcal{D}_W^*) \leq d_{\partial W^*}(D_0, E) \leq d_{\partial W^*}(D_0, \gamma) + d_{\partial W^*}(\gamma, E) = 1 + 1 = 2$, a contradiction.

Claim 3.10. For each $D \subset \mathcal{D}_2$, $E \subset \mathcal{D}^3$, $D \cap E \neq \emptyset$.

In this case, D is essential in V^* and $E \cap E_0 \neq \emptyset$. Suppose to the contrary that $D \cap E = \emptyset$.

By Lemma 2.2, there is an outermost arc γ_2 of ∂E and an outermost disk of E , say E_{γ_2} which is essential in W^* . Without loss of generality, we may assume that $\partial\gamma_2 \subset \partial E_2$.

Since $D \subset \mathcal{D}_2$, by the argument mentioned in Claim 3.9, there is a sub-arc γ_1 of ∂D such that $\partial\gamma_1 \subset \partial E_1$ and γ_1 together with an arc of E_1 forms an essential closed curve γ on ∂W^* such that $D_0 \cap \gamma = \emptyset$.

$D \cap E = \emptyset$ implies that $\gamma_1 \cap \gamma_2 = \emptyset$. According to our choice, $\partial\gamma_1$ and $\partial\gamma_2$ lie on distinct disks. Hence $E_{\gamma_2} \cap \gamma = \emptyset$. Thus

$$d_{\partial W^*}(f(\alpha), \mathcal{D}_W^*) \leq d_{\partial W^*}(D_0, E_{\gamma_2}) \leq d_{\partial W^*}(D_0, \gamma) + d_{\partial W^*}(\gamma, E_{\gamma_2}) = 2$$

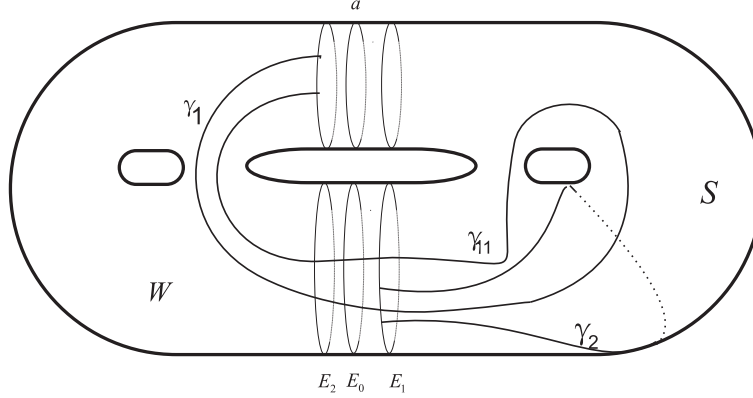
a contradiction.

Claim 3.11. For each $D \subset \mathcal{D}_3$, $E \subset \mathcal{D}^3$, $D \cap E \neq \emptyset$.

In this case, $D \cap D_0 \neq \emptyset$ and $E \cap E_0 \neq \emptyset$. Suppose to the contrary that $D \cap E = \emptyset$.

By Lemma 2.2, there exists an outermost arc γ_1 of ∂D and an outermost disk of D , say D_{γ_1} which is essential in V^* . Since $D_{\gamma_1} \cap \beta \neq \emptyset$ and β separates ∂V^* , there is an outermost arc say of γ_{11} such that $\gamma_{11} \subset \gamma_1$, moreover, $\partial\gamma_{11} \subset \partial E_1$ or $\partial\gamma_{11} \subset \partial E_2$. See Figure 2.

Without loss of generality, we may assume that $\partial\gamma_{11} \subset \partial E_1$. Thus γ_{11} together with an arc of E_1 forms an closed curve, say γ , which is essential in ∂W^* . We remark that $\gamma = \gamma_{11} \cup e_1$, where e_1 is an arc in E_1 . Since the outermost arc of D and E_2 are disjoint from α , $D_0 \cap \gamma = \emptyset$.

FIGURE 2. outermost arcs in $S \setminus N(\beta)$

By applying Lemma 2.2 to W and W^* , there is an outermost arc of ∂E , say γ_2 , where $\partial\gamma_2 \subset \partial E_1$ or $\partial\gamma_2 \subset \partial E_2$.

$D \cap E = \emptyset$ means that γ_{11} and γ_2 are disjoint.

If $\partial\gamma_2 \subset \partial E_2$, γ_2 , together with an arc in E_2 , bounds an outermost disk, say E_{γ_2} , which is essential in ∂W^* . In this case γ and E_{γ_2} are disjoint. It follows that:

$$d_{\partial W^*}(f(\alpha), \mathcal{D}_W^*) \leq d_{\partial W^*}(D_0, E_{\gamma_2}) \leq d_{\partial W^*}(D_0, \gamma) + d_{\partial W^*}(\gamma, E_{\gamma_2}) = 2$$

which is a contradiction.

If $\partial\gamma_2 \subset \partial E_1$, γ_2 , together with an arc e_2 in E_1 , bounds an outermost disk, say E_{γ_2} , which is essential in ∂W^* . In this case γ intersects E_{γ_2} in at most one point, since e_1 intersects e_2 at most one point. It follows that:

$$d_{\partial W^*}(f(\alpha), \mathcal{D}_W^*) \leq d_{\partial W^*}(D_0, E_{\gamma_2}) \leq d_{\partial W^*}(D_0, \gamma) + d_{\partial W^*}(\gamma, E_{\gamma_2}) \leq 1 + 2 = 3$$

which is also a contradiction.

If for some (i, j) , each $D \subset \mathcal{D}_i$, $E \subset \mathcal{D}_j$, $D \cap E = \partial D \cap \partial E \neq \emptyset$ holds, then by the symmetric construction of the Heegaard splitting and the same partitions of the \mathcal{D}_V and \mathcal{D}_W , it also holds for (j, i) . This completes the proof. \square

4. ON TOPOLOGICALLY MINIMAL AMALGAMATED HEEGAARD SURFACES

Let N be a 3-manifold. Suppose F is an incompressible separating surface which cuts N into N_1 and N_2 . Then N is obtained by gluing two 3-manifolds, say N_1 and N_2 , along a homeomorphism f . If N_i admits a Heegaard splitting $V_i \cup_{S_i} W_i$ for $i = 1, 2$, N has a natural Heegaard splitting called the amalgamation of $V_i \cup_{S_i} W_i$ for $i = 1, 2$, as:

$$N = C_1 \cup_S C_2 = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2)$$

The amalgamation of two unstabilized Heegaard splittings may be stabilized, see[13]. However, if the gluing map f is complicated enough, then the amalgamation of two minimal Heegaard splittings is unstabilized[11]. On the other hand, the amalgamation of two high distance Heegaard splittings is unstabilized[14].

David Bachman showed that if the gluing map f is complicated enough, then the amalgamated Heegaard surface is not topologically minimal[2]. We apply the idea of [14] to conjecture that the amalgamated Heegaard surface of two high distance Heegaard splittings is not topologically minimal. Our construction of keen weakly reducible Heegaard splittings presents positive examples.

Theorem 4.1. *Let M be a 3-manifold. Suppose that M admits an amalgamated Heegaard splitting:*

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2).$$

such that $g(F) = g(S_1) = g(S_2)$ and $d(S_i) \geq 4$, for $i = 1, 2$, then $M = V \cup_S W$ is keen weakly reducible hence S is not topologically minimal.

Proof. For each i , $d(S_i) \geq 4$ implies that $V_i \cup_{S_i} W_i$ is not an trivial Heegaard splitting. Since $g(\partial_+ W_i) = g(S_i) = g(F) = g(\partial_- W_i)$, F is disconnected and there exists only one separating compression disk embedded in W_i . (See figure 3) If $d(S_i) \geq 4$, it is easy to check that $V \cup_S W$ coincides with our construction. \square

We remark that if $d(S_i) \geq 2$ then S may not be keen and be topologically minimal. One example is that the standard Heegaard surface of $F^* \times S^1$, where $F^* \times S^1$ is a S^1 -bundle of a connected closed surface. In this case $V_i \cup_{S_i} W_i$ is the standard type 2 Heegaard splitting of $F^* \times I$ and $d(S_i) = 2$. the standard Heegaard surface of $F^* \times S^1$ is not keen and critical. In fact, if $d(S_i) \geq 2$ we show no higher index topologically minimal Heegaard surfaces exist.

Theorem 4.2. *Let M be a 3-manifold. Suppose that M admits an amalgamated Heegaard splitting:*

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_F (W_2 \cup_{S_2} V_2).$$

such that $g(F) = g(S_1) = g(S_2)$ and $d(S_i) \geq 2$, for $i = 1, 2$, then S is either critical or not topologically minimal.

Proof. We use notions as defined in Section 3. $d(S_i) \geq 2$ means that $d_{\partial V^*}(\beta, \mathcal{D}_{V^*}) \geq 2$ and $d_{\partial W^*}(f(\alpha), \mathcal{D}_{W^*}) \geq 2$. By Claim 3.2, Claim 3.3 and Claim 3.5, if $d(S_i) \geq 2$, any disk in $\mathcal{D}_W \setminus E_0$ intersects D_0 , equally, any disk

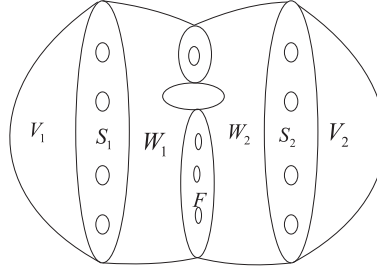


FIGURE 3. amalgamation of two Heegaard splittings

in $\mathcal{D}_V \setminus D_0$ intersects E_0 . Hence if there exist $D \subset \mathcal{D}_V \setminus D_0$ and $E \subset \mathcal{D}_W \setminus E_0$, such that $D \cap E = \emptyset$ then $\pi_1(\mathcal{D}(S)) \neq 1$ and S is a critical Heegaard surface. if no such pair of disks exists, S is keen weakly reducible and not topologically minimal. \square

REFERENCES

1. D. Bachman, Critical Heegaard surfaces, Trans. Amer. Math. Soc. 354(2002), 4015-4042.
2. D. Bachman, Stabilizing and destabilizing Heegaard splittings of suciently complicated 3-manifolds. Mathematische Annalen, 2013,355(2): 697-728.
3. D. Bachman, Connected sums of unstabilized Heegaard splittings are unstabilized, Beom. Topol. 12 (2008), 2327-2378.
4. D. Bachman and J. Johnson, On the existence of high index topologically minimal surfaces, Math. Res. Lett. 17 (2010), 389-394.
5. D. Bachman, Topological index theory for surfaces in 3-manifolds, Geom. Topol. 14 (2010), 585-609.
6. A.J.Casson, C.McA.Gordon, Reducing Heegaard splittings,Topology Appl. 27(1987) 275-283.
7. Q. E and F. Lei, Critical Heegaard surfaces obtained by self-amalgamation, Journal of Knot Theory and Its Ramifications, 22(05).
8. Q. E and F. Lei, Topologically minimal surfaces versus self-amalgamated Heegaard surfaces[J]. Science China Mathematics, 2014, 57(11):2393-2398.
9. J.Hempel, 3-manifolds as viewed from the curve complex, Topology 40(2001) 631-657.
10. A. Ido, Y. Jang and T. Kobayashi, on keen Heegaard splittings, preprint. Available at arxiv.org/abs/1605.04513.
11. M. Lackenby, The Heegaard genus of amalgamated 3-manifolds, Geom. Dedicata 109 (2004) 139C145.
12. D. McCullough. Virtually geometrically finite mapping class groups of 3-manifolds. J. Differential Geom, 33(1991), no.1: 1-65.
13. T. Kobayashi, R. Qiu, Y. Rieck and S. Wang, Separating incompressible surfaces and stabilizations of Heegaard splittings, Math. Proc. Cambridge Philos. Soc. 137 (2004) 633C643.
14. T. Kobayashi, R. Qiu, The amalgamation of high distance Heegaard splittings is always efficient, Math. Ann. 341 (2008) 707C715.

15. Y. Zou; K. Du; Q. Guo and R. Qiu; Unstabilized self-amalgamation of a Heegaard splitting. *Topology Appl.* 160 (2013), no. 2, 406-411.

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